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Estimation of Transfer Functions Using the Fourier Transform Ratio Method

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The advent of fast Fourier transform computational methods such as the Cooley-Tukey fast Fourier transform digital computer algorithm and coherent optical Fourier transform techniques have made computationally feasible the well-known Fourier transform ratio method of estimating transfer functions. This method estimates the transfer function by taking the ratio of the Fourier transform of finite length samples of input and response measurements. First the probability density of the error term for this method is derived in the case where the input and response are looked at as sample functions of stationary stochastic processes. Then the probability density of the error term is derived in the case where the input and response are transient in nature and hence must be considered as deterministic processes. An averaging method is then suggested to reduce the error term and a means for obtaining the probability that the error term is within certain limits is indicated.

Nomenclature

a_i = Fourier coefficient
α^2 = variance of a_i and b_i
b_i = Fourier coefficient
C = maximum value of the error term
$D(\tau)$ = lag window
$E_T(\omega)$ = error term in the estimate $\hat{H}_2(\omega)$ where the input and response signals are T sec long
$E'_T(\omega)$ = error term in the stationary case due to random noise
$E''_T(\omega)$ = error term in the stationary case due to systematic noise

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$E'_{T,t}(\omega)$ = random component of the error term in the case of transient inputs
$E''_{T,t}(\omega)$ = systematic component of the error term in the case of transient inputs
$\bar{E}_T(\omega)$ = portion of the error term which is a random variable
K = constant to adjust the probability density of $E'_T(\omega)$ to account for the fact that $E'_T(\omega)$ is bounded
$H(\omega)$ = transfer function
$H_i(\omega)$ = transfer function relating $x_i(t)$ to $z_i(t)$ in a multiple input linear system
$\hat{H}_{2,i}(\omega)$ = Fourier transform ratio method estimate of $H_i(\omega)$
$\hat{H}_1(\omega)$ = estimate of $H(\omega)$ equal to $S_{zx}(\omega)/S_x(\omega) + S_{nx}(\omega)/S_x(\omega)$
$\hat{H}_2(\omega)$ = estimate of $H(\omega)$ equal to $Z_T(\omega)/X_T(\omega) + E_T(\omega)$
$n(t)$ = noise in the measured data
$n'(t)$ = random noise
$n''(t)$ = systematic noise
$N_T(\omega)$ = Fourier transform of a sample function of $n(t)$ which is T sec in duration
$N'_T(\omega)$ = Fourier transform of a sample function of $n'(t)$ which is T sec long
$N''_T(\omega)$ = Fourier transform of a portion of $n''(t)$ which is T sec long
ω = angular frequency
ω_0 = fundamental frequency of the Fourier expansion of a portion T sec long of a stochastic process $(2\pi/T)$
$Q_{n'}$ = area under the autocorrelation function of $n'(t)$
Q_x = area under the autocorrelation function of $x(t)$
$R_x(\tau)$ = estimate of the autocorrelation of a stochastic process $x(t)$
$R_{zx}(\tau)$ = estimate of the cross correlation of stochastic processes $z(t)$ and $x(t)$
$S_x(\omega)$ = estimate of the power spectrum of $x(t)$
$S_{zx}(\omega)$ = estimate of the cross spectrum of $z(t)$ and $x(t)$
$\bar{\sigma}^2$ = variance of $\bar{E}_T(\omega)$
σ'^2 = variance of $E'_T(\omega)$
σ''^2 = variance of $E''_T(\omega)$
τ,t = time variables
T = sample length
T_m = length of lag window $D(\tau)$
$x(t)$ = stochastic process representing wind gust velocities
$x_i(t)$ = input entering the ith port of a multiple input linear system in the stationary case
$X_{T,i}(\omega)$ = Fourier transform of the input entering the ith port in the stationary case
$X_{T,t,i}(\omega)$ = Fourier transform of the input entering the ith port in the transient case
$X_T(\omega)$ = Fourier transform of a portion of a sample function of $x(t)$ which is T sec long
W = highest frequency at which an estimate of $H(\omega)$ is desired
$W(\omega)$ = weighting function
$z(t)$ = stochastic process representing strains in the aircraft caused by $x(t)$
$z_i(t)$ = output due to $x_i(t)$
$Z_{T,i}(\omega)$ = Fourier transform of the output due to the input of the ith port in the stationary case
$Z_{T,t}(\omega)$ = Fourier transform of the output in the transient case
$Z_{T,t,i}(\omega)$ = Fourier transform of the output produced by the ith input in the transient case
$Z_T(\omega)$ = Fourier transform of a portion of $z(t)$ which is T sec long

Introduction

A PROBLEM commonly occurring in the aircraft industry is that of obtaining the transfer function relating the strains in an aircraft structure to the wind gust or disturbance producing these strains. Time records of the disturbances $x(t)$ and strains $z(t)$ are made. These time records are then mathematically operated on to yield the transfer function $H(\omega)$ relating the "input" gusts $x(t)$ to the "response" strains $z(t)$. When an aircraft is in flight and experiencing atmospheric turbulence, $x(t)$ and $z(t)$ are considered to be sample functions of stationary stochastic processes. In this case, the cross-spectrum method is the most widely used method to estimate transfer functions. Here the power spectrum of $x(t)$ and the cross spectrum of $x(t)$ with the response $z(t)$ are estimated and their ratio is taken to form the estimate of the transfer function $H(\omega)$. When $x(t)$ and $z(t)$ are transient in nature, i.e., naturally exist over a finite time interval, they can no longer be considered stationary stochastic processes. An aircraft subjected to sudden transient gusts which last for a finite length of time would be an example. In this nonstationary case, the concepts of power spectra and cross spectra become undefined. Hence another method must be found to estimate $H(\omega)$.

The well-known Fourier transform ratio method estimates the transfer function $H(\omega)$ as the ratio of the Fourier transforms of $x(t)$ and $z(t)$. This method is applicable in both the case where $x(t)$ and $z(t)$ are stationary stochastic processes and when $x(t)$ and $z(t)$ are transient. Prior to the introduction of fast Fourier transform techniques,[1,2] this method was used only in cases where the sample lengths were very short, due to the large amount of computer time required. However, with the fast Fourier transform, both the cross-spectrum and the Fourier transform ratio methods should take approximately the same amount of computer time.

This paper presents a derivation of the probability density of the error term incurred in the estimation of $H(\omega)$ by the Fourier transform ratio method. This error term is derived in both the case when $x(t)$ and $z(t)$ are considered to be sample functions of stationary stochastic processes and when $x(t)$ and $z(t)$ are transient. An averaging technique is then suggested to reduce the error in both cases. A means to estimate the size of the error is indicated.

Estimation of Transfer Functions Using the Cross-Spectrum Method

The cross-spectrum method, although discussed extensively in the literature,[3,5] will be reviewed here. Throughout the discussion, it is assumed that $H(\omega)$ is not a time varying system. Both $x(t)$ and $z(t)$ are considered to be portions T sec long of stationary stochastic processes. An estimate $S_{zx}(\omega)$ of the cross spectrum of $z(t)$ and $x(t)$ and an estimate $S_x(\omega)$ of the power spectrum of $x(t)$ are made. Their ratio is then taken to form the estimate

$$H(\omega) = S_{zx}(\omega)/S_x(\omega) \quad (1)$$

Before the introduction of fast Fourier transform methods, the cross spectrum was estimated by computing on estimate $R_{zx}(\tau)$ of the cross correlation of $z(t)$ and $x(t)$

$$R_{zx}(\tau) = \frac{1}{T-|\tau|}\int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} z\left(t+\frac{\tau}{2}\right)x\left(t-\frac{\tau}{2}\right)dt \quad (2)$$

This estimate is then multiplied by an appropriate lag window $D(\tau)$ to yield the product $D(\tau)R_{zx}(\tau)$. Since $D(\tau)$ is generally nonzero for only $|\tau| \leq T_m$ where T_m is 5 to 10% of T, $D(\tau)$ effectively truncates $R_{zx}(\tau)$. A Fourier transform is then taken of $D(\tau)R_{zx}(\tau)$ to yield $S_{zx}(\omega)$;

$$S_{zx}(\omega) = \int_{-T_m}^{T_m} D(\tau)R_{zx}(\tau)e^{-j\omega\tau}\,d\tau \quad (3)$$

In a similar manner, an estimate $S_x(\omega)$ of the power spectrum of $x(t)$ is made by making an estimate of the autocorrelation

$$R_x(\tau) = \frac{1}{T-|\tau|}\int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} x\left(t+\frac{\tau}{2}\right)x\left(t-\frac{\tau}{2}\right)dt \quad (4)$$

and multiplying $R_x(\tau)$ by $D(\tau)$. A Fourier transform is then taken of the resulting product to yield $S_x(\omega)$ an estimate of the power spectrum of $x(t)$ (Ref. 4);

$$S_x(\omega) = \int_{-T_m}^{T_m} R_x(\tau)D(\tau)e^{-j\omega\tau}\,d\tau \quad (5)$$

The window function $D(\tau)$ effectively truncates both $R_{zx}(\tau)$ and $R_x(\tau)$ to a fraction of their original length. Since the products $R_x(\tau)D(\tau)$ and $R_{zx}(\tau)D(\tau)$ must be Fourier trans-

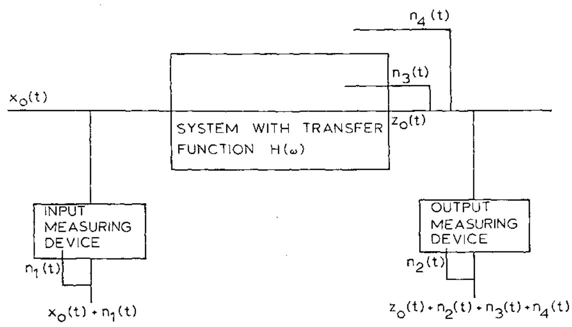

Fig. 1 The "true input" $x_0(t)$ produces the output $z_0(t)$. Possible sources of noise are $n_1(t)$ error in the input measuring instrument, $n_2(t)$ error in the output measuring instrument, $n_3(t)$ internally generated noise, and $n_4(t)$ unobserved noise from a path different than $x_0(t)$.

formed, and the Fourier transform operation is a lengthy one to perform by conventional digital computer techniques, this effective truncation greatly reduced the required computer time.

Using the fast Fourier transform, the cross-spectrum method is implemented by computing the Fourier transforms $X_T(\omega)$ of $x(t)$ and $Z_T(\omega)$ of $z(t)$ as follows:

$$X_T(\omega) = \int_{-T/2}^{T/2} x(t)e^{-j\omega t}\,dt \tag{6}$$

$$Z_T(\omega) = \int_{-T/2}^{T/2} z(t)e^{-j\omega t}\,dt \tag{7}$$

If $H(\omega)$ is to be evaluated from d.c. to Wcps and $x(t)$ and $z(t)$ are measured for T sec, there will be TW estimates of $H(\omega)$ available at frequency intervals of $2\pi/T$. The power spectrum of $x(t)$ and the cross spectrum of $z(t)$ and $x(t)$ are estimated, respectively, as

$$\begin{aligned} S_x(\omega) &= \frac{1}{2N}\sum_{i=-N}^{N} X_T\left(\omega + \frac{2\pi i}{T}\right) X^*_T\left(\omega + \frac{2\pi i}{T}\right) \times W\left(\omega + \frac{2\pi i}{T}\right) \\ S_{zx}(\omega) &= \frac{1}{2N}\sum_{i=-N}^{N} X^*_T\left(\omega + \frac{2\pi i}{T}\right) Z_T\left(\omega + \frac{2\pi i}{T}\right) \times W\left(\omega + \frac{2\pi i}{T}\right) \end{aligned} \tag{8}$$

$W(\omega)$ and N play a role similar to $D(\tau)$ in that they also adjust the statistics of the estimates of power spectrum and cross spectrum.

Thus far, noise in the input and response signals has not been considered. In practice, noise does exist and there are several likely sources of this noise. Clearly, part of the response $z(t)$ may be produced by other sources than the one designated as "input" $x(t)$. Strains can be produced in the aircraft by forces other than wind gusts. Changes in the flight controls to keep the aircraft on course can produce strains. Errors in instrument readings can add noise. Hence, the response is only partly due to $x(t)$. The rest results from spurious, unobserved inputs or internal noise. The sources of this noise are outlined in Fig. 1. As shown in Fig. 2, the portion of the observed output which is due to the measured input $x(t)$ is designated as $z(t)$ and the portion of the output due to any other source is designated $n(t)$. In other words, there is no loss in generality in considering all the noise in the output. Thus the time functions one has to operate on are $x(t)$ and $z(t) + n(t)$. Note from Fig. 2 that if $n_1(t) \neq 0$, neither $x(t)$ or $z(t)$ can be statistically independent of $n(t)$ since $x(t)$ and $z(t)$ have a component due to $n_1(t)$. If the transfer function is estimated by taking the ratio of the cross spectrum of the output with the input and the power spectrum of the input when noise is present, an estimate $\hat{H}_1(\omega)$ of the transfer function is obtained;

$$\begin{aligned} \hat{H}_1(\omega) &= S_{zx}(\omega)/S_x(\omega) + S_{nx}(\omega)/S_x(\omega) \\ &= H(\omega) + S_{nx}(\omega)/S_x(\omega) \end{aligned} \tag{9}$$

where $S_{nx}(\omega)$ is an estimate of the cross spectrum of $n(t)$ and $x(t)$. The mean value of the error term $S_{nx}/S_x(\omega)$ can be shown to be zero if $x(t)$ and $n(t)$ are statistically independent. Furthermore, the variance of the error term can be reduced to an arbitrarily small value as T becomes infinite if $S_{nx}(\omega)$ is averaged over a finite bandwidth $\Delta\omega$. The variance is not reduced if $S_{nx}(\omega)$ is not averaged over some $\Delta\omega$.[3] An indication of the magnitude of the error term $S_{nx}(\omega)/S_x(\omega)$ can be obtained by computing a function called the "coherence function" defined as[6]

$$\gamma_{x(z+n)}^2(\omega) = |S_{x(z+n)}(\omega)|^2/|S_x(\omega)||S_{z+n}(\omega)| \tag{10}$$

This function indicates the amount of noise in the data. A low value of $\gamma^2_{x(z+n)}$ would mean a large amount of noise and a high value would indicate a small amount of noise. $\gamma^2_{x(z+n)}$ can be used to calculate quantitative estimates of the magnitude and phase of the error term $S_{x(z+n)}(\omega)/S_x(\omega)$.

As the size of the aircraft becomes large relative to the scale of wind gust velocities, a multiple input model becomes necessary to analyze the structure. In this case, the system has inputs $x_i(t)$, $i = 1, \ldots, N$, and a single output $z(t)$. The problem is to estimate $H_i(\omega)$, the transfer function relating $x_i(t)$ to the portion of $z(t)$ which it produced. The cross-spectrum method has been extended to this case and "multiple coherence functions" have been defined to estimate the error incurred.[3,6]

Estimation of Transfer Functions Using the Fourier Transform Ratio Method

To estimate the transfer function $H(\omega)$ by the Fourier transform ratio method, $Z_T(\omega)$, the Fourier transform of $z(t)$ and $X_T(\omega)$ the Fourier transform of $x(t)$ are computed. Their ratio is then taken to form

$$H(\omega) = Z_T(\omega)/X_T(\omega) \tag{11}$$

which yields an estimate of $H(\omega)$ at each value of ω for which $X(\omega) \neq 0$. If $H(\omega)$ is to be evaluated from d.c. to Wcps and $x(t)$ and $z(t)$ are measured for T sec, there will be TW estimates of $H(\omega)$ available at frequency intervals of $2\pi/T$. In the case where noise is present, an estimate $\hat{H}_2(\omega)$ is formed;

$$\begin{aligned} \hat{H}_2(\omega) &= H(\omega) + N_T(\omega)/X_T(\omega) \\ &= H(\omega) + E_T(\omega) \end{aligned} \tag{12}$$

where $E_T(\omega) = N_T(\omega)/X_T(\omega)$.

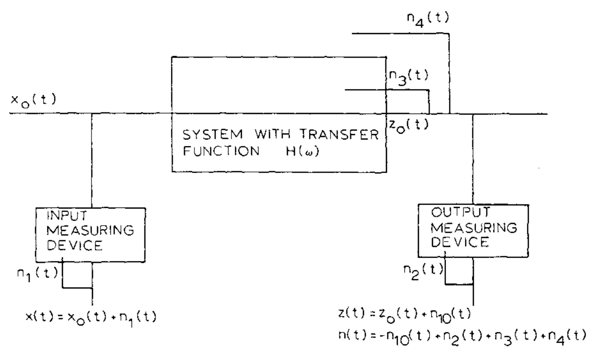

Fig. 2 Equivalent case in which all noise is referred to the output; here $x(t) = x_0(t) + n_1(t)$ is considered the input and $z(t) = z_0(t) + n_{10}(t)$ is considered the output where $n_{10}(t)$ is the portion of the output produced by $n_1(t)$. The noise term in the output is then $n(t) = -n_{10}(t) + n_2(t) + n_3(t) + n_4(t)$.

One past objection to this method, namely large amounts of computer time required to compute Fourier transforms of samples T sec long rather than 5% to 10% of T as in the older method, is alleviated by the aforementioned fast Fourier transform techniques. The other objection is that the error term $E_T(\omega)$ is not necessarily small.[7] This objection is a valid one, but detailed analyses of the statistics of the error term will show that it can be reduced by averaging the estimate $\hat{H}_2(\omega)$ over a suitable frequency interval $\Delta\omega$. In fact, averaging $\hat{H}_2(\omega)$ over an interval $\Delta\omega$ will make the error arbitrarily small if T is arbitrarily large. The error term will be investigated in the cases where $x(t)$ and $z(t)$ are stationary stochastic processes and where $x(t)$ and $z(t)$ are transient. Random errors and systematic errors will be discussed in both cases. The noise $n(t)$ can have a random component $n'(t)$ and a systematic component $n''(t)$. This will result in an error term

$$\begin{aligned} E_T(\omega) &= N_T(\omega)/X_T(\omega) \\ &= N'_T(\omega)/X_T(\omega) + N''_T(\omega)/X_T(\omega) \end{aligned} \tag{13}$$

Both kinds of error will be analyzed separately.

Statistics of the Error Term When $x(t)$ and $z(t)$ are Stationary

Both $x(t)$ and $z(t)$ will now be considered to be normal stationary stochastic processes. To analyze the case where $n'(t)$ is random noise, $n'(t)$ will be assumed to be a stationary normal stochastic process of zero mean which is statistically independent of $x(t)$ and $z(t)$. It is shown in Appendix A that the probability densities of $X_T(\omega)$ and $N'_T(\omega)$ are Rayleigh distributed in amplitude and uniformly distributed in phase;

$$p[|X_T(\omega)|] = \frac{2|X_T(\omega)|}{Q_xT}\, e^{-|X_T(\omega)|^2/Q_xT} \tag{14}$$

$$p[\angle X_T(\omega)] = 1/2\pi$$

$$p[|N'_T(\omega)|] = \frac{2|N'_T(\omega)|}{Q_{n'}T}\, e^{-|N'_T(\omega)|^2/Q_{n'}T} \tag{15}$$

$$p[\angle N'_T(\omega)] = 1/2\pi$$

for $\omega \leq 2\pi W$ where W is the upper limit of the frequency range over which $H(\omega)$ is to be estimated. $Q_{n'}$ and Q_x are the areas under the autocorrelation functions of $n'(t)$ and $x(t)$, respectively.

Note that the probability density of $X_T(\omega)$ and $N'_T(\omega)$ is a function of the sample length T. As T increases, the variance of both $X_T(\omega)$ and $N'_T(\omega)$ will increase in the same manner. Hence, both $X_T(\omega)$ and $N'_T(\omega)$ will "grow" at the same "rate" as T increases. $N'_T(\omega)$ is now divided by $X_T(\omega)$ to form the error term $E'_T(\omega)$. It is shown in Appendix B that the probability density of the amplitude of $E'_T(\omega)$ is

$$p[|E'_T(\omega)|] = \frac{KQ_{n'}|E'_T(\omega)|}{2Q_x[|E'_T(\omega)|^2/2 + Q_{n'}/2Q_x]^2} \tag{16}$$

$$\text{for } |E'_T(\omega)| \leq C$$

$$p[|E'_T(\omega)|] = 0 \text{ for } |E'_T(\omega)| > C$$

and that the probability density of the phase is

$$p[\angle E'_T(\omega)] = 1/2\pi$$

The fact that $E'_T(\omega)$ is always less than a constant C stems from practical considerations. The only way by which $E'_T(\omega)$ can become arbitrarily large is for $N'_T(\omega)$ to be arbitrarily large or $X_T(\omega)$ to become arbitrarily small. If $N'_T(\omega)$ is arbitrarily large for a particular ω, then $n'(t)$ has some arbitrarily large frequency component. If this is the case, the dynamic range of the recording device would limit the observed magnitude of $n'(t)$. On the other hand, if $X_T(\omega)$ is very small or zero at a particular ω, then effectively there is no input at ω and no estimate of $H(\omega)$ at ω can be made. Hence $E'_T(\omega)$ will be bounded since $|N'_T(\omega)|$ is limited by the dynamic range of the recording medium and any estimate of $H(\omega)$ is disregarded at values of ω for which $X_T(\omega)$ is small. The constant K adjusts the probability density to take this effective truncation of $|E'_T(\omega)|$ into account. Since it is not important to the following discussion, no attempt will be made to estimate K and C. Since $E'_T(\omega)$ is uniformly distributed in phase, its mean value is zero as shown in Appendix B. The placing of an upper bound on $|E'_T(\omega)|$ has effectively given $E'_T(\omega)$ a variance, say σ'^2.

Note that the probability density of $E'_T(\omega)$ is independent of T. However, since the probability densities of $N'_T(\omega)$ and $X_T(\omega)$ indicate that $N'_T(\omega)$ and $X_T(\omega)$ "grow" in the same manner as T increases, one might expect their ratio to be independent of T. The probability density of $E'_T(\omega)$ being independent of T indicates that if $H(\omega)$ is estimated by taking the ratio of $Z_T(\omega) + N'_T(\omega)$ to $X_T(\omega)$, nothing is gained in error reduction by taking longer samples so long as T is long compared to the memory of the system. No use is made of the additional information provided by increased sample length. However, it will be shown that if $\hat{H}_2(\omega)$ is averaged over a suitable interval $\Delta\omega$, the error term will decrease as T increases.

Now the error term $E''_T(\omega)$ due to the systematic noise will be analyzed. $E''_T(\omega)$ is given by

$$E''_T(\omega) = N''_T(\omega)/X_T(\omega) \tag{17}$$

where $N''_T(\omega)$ is the Fourier transform of $n''(t)$, the systematic component of the noise. The systematic noise can be due to several sources such as calibration errors in the measuring instruments. Since no statistical information is available for a systematic error, $n''(t)$ will be considered to be a deterministic process. Hence $E''_T(\omega)$ is the ratio of a deterministic function divided by a random variable which makes $E''_T(\omega)$ a random variable. The probability density of $E''_T(\omega)$ is given by

$$p(|E''_T(\omega)|) = \frac{2|N''_T(\omega)|^2}{Q_xT|E''_T(\omega)|^3} \exp\left[\frac{-|N''_T(\omega)|^2}{Q_xT|E''_T(\omega)|^2}\right] \tag{18}$$

and $p[\angle E''_T(\omega)] = 1/2\pi$.

The uniformity in phase of $E''_T(\omega)$ makes its mean value zero. In the strict sense, $E''_T(\omega)$ has no variance. However, since small values of $X_T(\omega)$ are disregarded, and the dynamic range of the recording medium limits the observed values of the noise, $E''_T(\omega)$ can be assigned variance σ''^2. These arguments are similar to those assigning a variance to $E'_T(\omega)$. The details of the derivation of the probability density of $E''_T(\omega)$, its mean and variance are given in Appendix C. In contrast to $E'_T(\omega)$, the probability density of $E''_T(\omega)$ is not independent of T. This follows from the fact that no probability density was assumed for $n''(t)$ as was assumed for $n'(t)$. Hence, $x(t)$ and $n''(t)$ need not grow in the same manner as T increases, and T would not cancel when their ratio is taken. It is shown in Appendix A, that the preceding expressions for the probability density of the error terms are only exactly true for values of ω which are multiples of $2\pi/T$.

Statistics of the Error Term When $x(t)$ and $z(t)$ are Transient

In the previous section, $x(t)$ and $z(t)$ were assumed to be stationary stochastic processes. This assumption is frequently used for aircraft in service experiencing atmospheric turbulence. However, in certain situations, such as loading by sudden wind gusts, $x(t)$ and $z(t)$ naturally exist for only a finite time period T. In no sense can $x(t)$ and $z(t)$ then be considered stationary. One approach in handling such problems is to consider $x(t)$ and $z(t)$ nonstationary processes whose probability of occurring outside of the interval $-T/2 \leq t \leq T/2$ is zero. This approach involves the very complex field of nonstationary processes and will not be attempted. Instead, no statistical information will be assumed for $x(t)$ and $z(t)$, i.e. they will be thought of as determinisitic but unknown processes. The noise term $n(t)$ will still be composed of a random term $n'(t)$ and a systematic term $n''(t)$. It will be seen that this restricted case yields useful results.

First the random component of the error term will be analyzed. This component is given by

$$E'_{T,t}(\omega) = N'_T(\omega)/X_{T,t}(\omega) \tag{19}$$

where the subscript t denotes the transient case. $N'_T(\omega)$ has been shown in Eq. 15 to be Rayleigh distributed in amplitude and uniformly distributed in phase. Since $1/X_{T,t}(\omega)$ is a deterministic coefficient of $N'_T(\omega)$ in the term $E'_{T,t}(\omega)$, the probability density of the amplitude of $E'_{T,t}(\omega)$ is also Rayleigh

$$p[|E'_{T,t}(\omega)|] = \frac{2|E'_{T,t}(\omega)||X_{T,t}(\omega)|^2}{Q_{n'}T} \exp\left[\frac{-|E'_{T,t}(\omega)|^2|X_{T,t}(\omega)|^2}{Q_{n'}T}\right] \tag{20}$$

and the phase of $E'_{T,t}(\omega)$ is uniform

$$p[\angle E'_{T,t}(\omega)] = 1/2\pi$$

Hence, $E'_{T,t}(\omega)$ has a zero mean and variance given by

$$\sigma'^2_{T,t} = (2 - \pi/2)Q_{n'}T/2X_{T,t}(\omega)|^2$$

In the transient case, the probability density of the random error term is dependent on T in contrast to the error term in the case of stationary inputs.

Little can be said about the component of the error term due to systematic error, namely

$$E''_{T,t}(\omega) = N''_T(\omega)/X_{T,t}(\omega) \tag{21}$$

No statistical information has been assumed for either $N''_T(\omega)$ or $X_{T,t}(\omega)$, so $E''_{T,t}(\omega)$ cannot be treated as a random variable. However, it will be shown that error reduction procedures described below will likely reduce this error whatever it turns out to be.

Reduction of the Error Term

The error reduction procedure described below will reduce the error incurred in the estimate $\hat{H}_2(\omega)$ in both the case of stationary and transient inputs. Quantitative estimates of the amount of error reduction can be obtained for the errors $E'_T(\omega)$, $E''_T(\omega)$, and $E'_{T,t}(\omega)$. Since the error term $E''_{T,t}(\omega)$ is not a random variable, no estimates will be made of the amount it is reduced. It will be clear, however, that this procedure will reduce the component of the error due to $E''_{T,t}(\omega)$. The error reduction procedure will be discussed in terms of $\bar{E}_T(\omega)$ which in the stationary case will include $E'_T(\omega)$ and $E''_T(\omega)$ and in the transient case will only $E'_{T,t}(\omega)$. Since the means of each of the components of $\bar{E}_T(\omega)$ are zero, the mean of $\bar{E}_T(\omega)$ is also zero. The variance of $\bar{E}_T(\omega)$ will be denoted by $\bar{\sigma}^2$.

It is shown in Appendix D that values of $\bar{E}_T(\omega)$ are statistically independent at multiples in frequency of $2\pi/T$. The reduction procedures below will only apply those frequencies which are multiples of $2\pi/T$. If T is large compared to the duration of the impulse response of the system, one would expect $H(\omega)$ to vary little over frequency intervals of $2\pi/T$. If the estimate $\hat{H}_2(\omega)$ is averaged over an interval $\Delta\omega$ which is large compared to $2\pi/T$, but not so large that $H(\omega)$ varies appreciably over $\Delta\omega$, the transfer function will be affected little by the averaging whereas $\bar{E}_T(\omega)$ will be reduced. For example, consider the following average of $\hat{H}_2(\omega)$ over $\Delta\omega$:

$$[\hat{H}_2(\omega)]_{\text{ave}} = \frac{1}{q}\sum_{i=-q/2}^{q/2} \hat{H}_2\left(\omega + \frac{2\pi i}{T}\right) = \frac{1}{q}\sum_{i=-q/2}^{q/2} H\left(\omega + \frac{2\pi i}{T}\right) + \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right) \tag{22}$$

where $q = \Delta\omega/2\pi/T = (\Delta\omega)T/2\pi$. Here, values of $\hat{H}_2(\omega)$ spaced $2\pi/T$ apart in an interval $\Delta\omega$ are averaged. Since $\Delta\omega$ has been chosen so that $H(\omega)$ varies little in this interval, it follows that

$$\frac{1}{q}\sum_{i=-q/2}^{q/2} H\left(\omega + \frac{2\pi i}{T}\right) \approx H(\omega) \tag{23}$$

Equation (13) now reduces to

$$[\hat{H}_2(\omega)]_{\text{ave}} = H(\omega) + \frac{1}{q}\sum_{i=-q/2}^{q/2} \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right) \tag{24}$$

The new error term

$$\frac{1}{q}\sum_{i=-q/2}^{q/2} \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right) \tag{25}$$

will have zero mean, since $\bar{E}_T(\omega)$ has zero mean. Since the values of $\bar{E}_T(\omega)$ are independent at frequency multiples of $2\pi/T$, the variance of the new error term is reduced by a factor $1/q$

$$\bar{\sigma}^2_{\text{ave}} = \left\langle\left|\frac{1}{q}\sum_{i=-q/2}^{q/2} \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right)\right|^2\right\rangle = \frac{\bar{\sigma}^2}{q} = \frac{\bar{\sigma}^2 2\pi}{T\Delta\omega} \tag{26}$$

where $\bar{\sigma}^2_{\text{ave}}$ is the variance of (25) and $\langle\rangle$ denotes the expected value.

In theory, the error term can be made arbitrarily small for T arbitrarily large. The variance also decreases as $1/\Delta\omega$. Hence it is desirable to make $\Delta\omega$ as large as possible so long as $H(\omega)$ varies little over $\Delta\omega$. Properly choosing $\Delta\omega$ requires an a priori knowledge of the amount of variation of $H(\omega)$ with $\Delta\omega$. However, for systems like aircraft structures, past experience from other aircraft will give one a "feeling" of how rapidly $H(\omega)$ fluctuates as ω varies so that a safe value of $\Delta\omega$ can be chosen.

The average of $\hat{H}_2(\omega)$ described previously was an unweighted average. The average, of course, could be a weighted average of the form

$$[\hat{H}_2(\omega)]_{\text{ave}} = \sum_{i=-q/2}^{q/2} H_2\left(\omega + \frac{2\pi i}{T}\right)W\left(\omega + \frac{2\pi i}{T}\right) \tag{27}$$

where $W(\omega)$ is a suitable weighting function which is nonzero only over a frequency interval $\Delta\omega$. Since $H(\omega)$ will vary some over $\Delta\omega$, it may be desirable to use a weighting function which gives more weight to values of the estimate $\hat{H}_2(\omega)$ close to the particular frequency at which $H(\omega)$ is being estimated. The amount of error reduction will vary with the weighting function used.

The error due to $E''_{T,t}(\omega)$ will likely be reduced by this averaging procedure. This procedure simply averages the error over an interval in the frequency domain and thereby reduces it. In general, averaging any complex function over an interval will reduce the value of that function due to cancellation of the various components during averaging. The more the phase of the function varies over the interval the greater will be the reduction. Since no information has been assumed about $E''_{T,t}(\omega)$ the amount of reduction cannot be estimated but it is likely it will be reduced somewhat by the procedure.

Confidence Levels

In order to assess the reliability of the estimates of the transfer function, it is necessary to assign confidence levels to the estimates of its amplitude and phase. If the second moment of the error term is known, the confidence levels can be determined from Chebyshev's Inequality which states[8]

$$p[|\bar{E}_T(\omega)| \leq \epsilon] \geq 1 - \langle|\bar{E}_T(\omega)|^2\rangle/\epsilon^2 \tag{28}$$

This expression will give a lower bound to the probability that the amplitude of $H(\omega)$ will be in the interval $[|\hat{H}_2(\omega)| - \epsilon, |\hat{H}_2(\omega)| + \epsilon]$ and that the phase of $H(\omega)$ will be in the interval $[\angle \hat{H}_2(\omega) - \tan^{-1} \epsilon/|\hat{H}_2(\omega)|, \hat{H}_2(\omega) + \tan^{-1} \epsilon/|\hat{H}_2(\omega)|]$. These intervals follow from the fact that the error term will have the greatest influence on the estimate of the amplitude of $H(\omega)$ when $\bar{E}_T(\omega)$ is 0° or 180° out of phase with $H(\omega)$. The error term will have the greatest influence on the phase estimate of $H(\omega)$ when the phase of $\bar{E}_T(\omega)$ is ±90° out of phase with $H(\omega)$.

It now remains to estimate the second moment $\langle|\bar{E}_T(\omega)|^2\rangle$ of the error term which is equal to $\bar{\sigma}^2$ since the mean of $\bar{E}_T(\omega)$ is zero. The second moment could be calculated directly from the probability density of $\bar{E}_T(\omega)$ if the constants Q_x, Q_n', K, and C were known. An easier, although indirect means to estimate the second moment is to form

$$\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle = \langle|[\bar{E}_T(\omega)]_{\text{ave}} - \bar{E}_T(\omega)|^2\rangle \tag{29}$$

where $\hat{H}_2(\omega)$ is the estimate of $H(\omega)$ before averaging and $[\hat{H}_2(\omega)]_{\text{ave}}$ is the estimate after averaging. For simplicity let $[\hat{H}_2(\omega)]_{\text{ave}}$ be the unweighted average of the estimate although similar results can be obtained for weighted averages. Then

$$[\bar{E}_T(\omega)]_{\text{ave}} = \frac{1}{q}\sum_{i=-q/2}^{q/2} \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right) \tag{30}$$

and (29) becomes

$$\begin{aligned}\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle &= \langle|[\bar{E}_T(\omega)]_{\text{ave}}|^2 - [\bar{E}_T(\omega)]_{\text{ave}} \times [\bar{E}_T(\omega)]^* - [\bar{E}_T(\omega)]^*_{\text{ave}}[\bar{E}_T(\omega)] + |[\bar{E}_T(\omega)]|^2\rangle \\ &= \frac{\bar{\sigma}^2}{q} - \frac{1}{q}\left\langle \sum_{i=-q/2}^{q/2} [\bar{E}_T(\omega)]^* \times \bar{E}_T\left(\omega + \frac{2\pi i}{T}\right)\right\rangle - \frac{1}{q}\left\langle \sum_{i=-q/2}^{q/2} \bar{E}_T(\omega) \times \left[\bar{E}_T\left(\omega + \frac{2\pi i}{T}\right)\right]^*\right\rangle + \bar{\sigma}^2\end{aligned} \tag{31}$$

Since $\bar{E}_T(\omega)$ is independent of $\bar{E}_T(\omega + 2\pi i/T)$ for $i \neq 0$ and ω *a* multiple of $2\pi/T$ and since both terms have zero mean, then (31) reduces to

$$\begin{aligned}\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle &= \frac{\bar{\sigma}^2}{q} - \frac{\bar{\sigma}^2}{q} - \frac{\bar{\sigma}^2}{q} + \bar{\sigma}^2 \\ &= \bar{\sigma}^2\left(1 - \frac{1}{q}\right)\end{aligned} \tag{32}$$

Hence $\bar{\sigma}^2$ is

$$\bar{\sigma}^2 = \frac{\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle}{1 - 1/q} \tag{33}$$

The variance $\bar{\sigma}^2$ has now been found in terms of $\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle$. The remaining problem is to estimate $\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle$. Since samples of $[\hat{H}_2(\omega)]_{\text{ave}}$ and $\hat{H}_2(\omega)$ are available at intervals in ω of $2\pi/T$, a reasonable estimate of $\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle$ is to average values of the difference $|(\hat{H}_2(\omega))_{\text{ave}} - \hat{H}_2(\omega)|^2$ at each frequency for which $[\hat{H}_2(\omega)]_{\text{ave}}$ and $\hat{H}_2(\omega)$ are available, forming the following estimate S of $\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle$:

$$\begin{aligned}S &= \frac{1}{TW}\sum_{i=1}^{TW}\left|\left[\hat{H}_2\left(\frac{2\pi i}{T}\right)\right]_{\text{ave}} - H_2\left(\frac{2\pi i}{T}\right)\right|^2 \\ &= \frac{1}{TW}\sum_{i=1}^{TW}\left|\left[\bar{E}\left(\frac{2\pi i}{T}\right)\right]_{\text{ave}} - \bar{E}_T\left(\frac{2\pi i}{T}\right)\right|^2\end{aligned}$$

If the statistics of $\bar{E}_T(\omega)$ are independent of ω and as is the case when $\bar{E}_T(\omega)$ is only composed of $E'_T(\omega)$, the limits of the summation can be 1 and TW as shown previously. The variance $\bar{\sigma}^2$ of $\bar{E}_T(\omega)$ might vary with ω in the stationary case when $E''_T(\omega)$ is nonzero or in the transient case. In this situation, the limits of the summation would have to such that the summation ranges only over the interval in frequency that $\bar{\sigma}^2$ is constant. In other words, several estimates would have to be made of $\bar{\sigma}^2$ over intervals in ω from zero to $2\pi W$. The estimates would have the form

$$S_j = \frac{1}{2T\Delta W}\sum_{i=-T\Delta W}^{T\Delta W}\left|\left[\bar{E}\left(\omega_j + \frac{2\pi i}{T}\right)\right]_{\text{ave}} - \bar{E}_T\left(\omega_j + \frac{2\pi i}{T}\right)\right|$$

where S_j is the estimate of σ_j^2, the average variance of the error term in an interval $2\pi\Delta W$ side centered at ω_j. Since values of $\bar{E}_T(\omega)$ have been shown to be statistically independent at multiples of $2\pi/T$, the mean of S_j can shown to be $\langle|[\hat{H}_2(\omega)]_{\text{ave}} - \hat{H}_2(\omega)|^2\rangle$ and the variance of S_j can be shown to decrease as $1/2T\Delta W$.[9] Similarly, the mean of S is the quantity we are estimating and the variance of S decreases as $1/TW$.

Multiple Input Linear Systems

As mentioned in the discussion of the cross-spectrum method, a multiple input linear system often becomes necessary to analyze an aircraft. Here, the problem is to find the transfer function $H_i(\omega)$ which relates the input $x_i(t)$ to the portion of the output $z(t)$ which it produced. In Refs. 3 and 6, the cross-spectrum method has been extended to this case.

So far, the Fourier transform ratio method has been discussed only for the case of a single input and single output. However, if the inputs $x_i(t)$ are statistically independent of each other, then the multiple stationary input case can be treated in the same manner as the single input case. In the case of multiple stationary inputs, an estimate $\hat{H}_{2,k}(\omega)$ of $H_k(\omega)$ is formed as

$$\hat{H}_{2,k}(\omega) = Z_T(\omega)/X_{T,k}(\omega) + E'_T(\omega) + E''_T(\omega) \tag{34}$$

where $X_{T,k}(\omega)$ is the Fourier transform of $x_k(t)$. $z(t)$ is the output due to all the inputs, and therefore

$$z(t) = \sum_{i=1}^{N} z_i(t)$$

and

$$Z_T(\omega) = \sum_{i=1}^{N} Z_{T,i}(\omega) \tag{35}$$

where $Z_{T,i}(\omega)$ is the Fourier transform of $z_i(t)$, the portion of the output due to $x_i(t)$. Substituting (35) into (34), $\hat{H}_{2,k}(\omega)$ can be written as

$$\hat{H}_{2,k}(\omega) = Z_{T,k}(\omega)/X_{T,k}(\omega) + \sum_{\substack{i=1\\ i\neq k}}^{N} Z_{T,i}(\omega)\Big/X_{T,k}(\omega) + E'_T(\omega) + E''_T(\omega) \tag{36}$$

$$\hat{H}_{2,k}(\omega) = H_k(\omega) + E'''_T(\omega) + E'_T(\omega) + E''_T(\omega)$$

where

$$E'''_T(\omega) = \sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{Z_{T,i}(\omega)}{X_{T,k}(\omega)} \tag{37}$$

In other words, when $H_k(\omega)$ is estimated, the outputs other than $z_k(t)$ act as noise terms. In the stationary case, if the inputs $x_i(t)$ are zero mean, independent, normal stochastic processes, the outputs $z_i(t)$ are also zero mean, independent, normal stochastic processes. Hence the statistics of

$$\sum_{\substack{i=1 \\ i\neq k}}^{N} Z_{T,i}(\omega) \tag{38}$$

are the same as $N'_T(\omega)$ which implies that $E'''_T(\omega)$ has the same statistics as $E'_T(\omega)$. Therefore, in the multiple stationary independent input case, the error term $E'''_T(\omega)$ can be considered a component of the error term $E'_T(\omega)$. Then the estimate $\hat{H}_{2,k}(\omega)$ takes the form

$$\hat{H}_{2,k}(\omega) = H_K(\omega) + E'_T(\omega) + E''_T(\omega) \tag{39}$$

where $E'_T(\omega)$ is amended to include $E'''_T(\omega)$. All of the error reduction and estimation operations discussed in the single input case can now be applied to the multiple input case when the inputs are statistically independent of each other.

The Fourier transform ratio method can be extended as indicated previously to the case of multiple inputs in the stationary input case when the inputs are statistically independent of each other. Further analysis is needed to extend this method to the case of correlated inputs. In contrast, the cross-spectrum method has been extended to the case of correlated multiple inputs as indicated in Refs. 3 and 6.

In the case of transient inputs, the multiple input case also reduces to the single input case. The estimate $\hat{H}_{2,k}(\omega)$ can be written as

$$\hat{H}_{2,k}(\omega) = Z_{T,t}(\omega)/X_{T,t,k}(\omega) + E'_{T,t}(\omega) + E''_{T,t}(\omega) \tag{40}$$

where $X_{T,t,k}(\omega)$ is the Fourier transform of the transient input $x_k(t)$. In analogy with the stationary input case, the total output is the sum of the outputs due to each input

$$z(t) = \sum_{i=1}^{N} z_i(t)$$

and (41)

$$Z_{T,t}(\omega) = \sum_{i=1}^{N} Z_{T,t,i}(\omega)$$

where $Z_{T,t,i}(\omega)$ is the Fourier transform of $z_i(t)$, the portion of the output due to $x_i(t)$. $\hat{H}_{2,k}(\omega)$ can now be written as

$$\hat{H}_{2,k}(\omega) = \frac{Z_{T,t,k}(\omega)}{X_{T,t,k}(\omega)} + \frac{\sum_{\substack{i=1 \\ i\neq k}}^{N} Z_{T,t,i}(\omega)}{X_{T,i,k}(\omega)} + E'_{T,t}(\omega) + E''_{T,t}(\omega)$$
$$= H_k(\omega) + E'''_{T,t}(\omega) + E'_{T,t}(\omega) + E''_{T,t}(\omega) \tag{42}$$

The new error term $E'''_{T,t}(\omega)$ is not a random variable since both its numerator

$$\sum_{\substack{i=1 \\ i\neq k}}^{N} Z_{T,t,k}(\omega)$$

and denominator $X_{T,i,k}(\omega)$ are deterministic. Hence $E'''_{T,t}(\omega)$ has the same character as $E''_{T,t}(\omega)$ and therefore can be lumped together with it. $\hat{H}_{2,k}(\omega)$ then takes the form

$$\hat{H}_{2,k}(\omega) = H_k(\omega) + E'_{T,t}(\omega) + E''_{T,t}(\omega)$$

where $E''_{T,t}(\omega)$ now includes $E'''_{T,t}(\omega)$.

Conclusions

The well-known Fourier transform ratio method of estimating transfer functions has been made computationally feasible for problems involving large quantities of data by fast Fourier transform techniques. This method provides a means of estimating transfer functions when the input and response measurements are either stationary stochastic processes or transient processes. In this paper, the error terms resulting in both cases have been derived. An averaging technique has been developed to reduce the error term and method presented to estimate the size of the error. The error reduction and estimation procedures are applicable in both the stationary and transient cases.

In the case of stationary inputs, the Fourier transform ratio method appears to be analogous to the more common cross-spectrum method. Both have error terms of zero mean value when the noise, referred to the output shown in Fig. 2, is statistically independent of input and response. Using the fast Fourier transform, both would likely take about the same amount of computer time. The window function $D(\tau)$ reduces the error through averaging in the cross-spectrum method in a manner similar to the error reduction procedure suggested for the Fourier transform ratio method. Both procedures have methods of estimating the size of the error. Both methods can estimate transfer functions for multiple input systems if the inputs are statistically independent. The cross-spectrum method has been developed in the case of correlated inputs whereas the Fourier transform ratio method has not. It is felt that the primary advantage of the cross-spectrum method over the Fourier transform ratio method for stationary inputs is that it has been further developed theoretically. The statistics of the Fourier transform ratio method appear somewhat simpler, however.

In the transient case, the concepts of power spectra and cross spectia are undefined and hence the cross-spectrum procedure is not applicable. This paper shows that the Fourier transform ratio method is a valid estimation procedure in this case. The mean value of the random error incurred is zero. The systematic error is difficult to treat and was only discussed qualitatively. The method was extended to the multiple input case, and the mean value of any normal stationary noise was shown to be zero. However, error due to other noise was only qualitatively treated.

Appendix A: Derivation of the Probability Density of $N'_T(\omega)$ and $X_T(\omega)$

Since similar assumptions have been made about the probability densities of $n'(t)$ and $x(t)$, namely that both are normal stochastic processes of zero mean, the form of the probability density of $X_T(\omega)$ will be the same as $N'_T(\omega)$. Therefore, the probability density of $N'_T(\omega)$ will be derived and the derivation of the probability density of $X_T(\omega)$ will be similar.

First of all, $n'(t)$ will be represented in the interval of measurement $(-T/2,T/2)$ by a Fourier series whose fundamental frequency is $\omega_0 = 2\pi/T$. Since the input and response are filtered to remove frequencies greater than Wcps, the portion of $n'(t)$ which is measured will only have frequency components up to Wcps and the series will only have $N + 1$ terms

$$n'(t) = \sum_{i=0}^{N} a_i \cos i\omega_0 t + b_i \sin i\omega_0 t \text{ for } |t| \leq T/2 \tag{A1}$$

where

$$a_i = \frac{2}{T}\int_{-T/2}^{T/2} n'(t) \cos i\omega_0 t dt$$

$$b_i = \frac{2}{T}\int_{-T/2}^{T/2} n'(t) \cos i\omega_0 t dt$$

$$N = 2\pi W/\omega_0$$

Since $n'(t)$ is normal with zero mean, it follows that a_i and b_i are gaussian with zero mean. Since we are only dealing with $n'(t)$ in the interval $(-T/2, T/2)$, let us consider $n'(t)$ to be periodic with a period of T. Then it follows that

$$\begin{aligned} \langle a_i a_j \rangle = \langle b_i b_j \rangle &= 0 \qquad \text{for } i \neq j \\ &= \alpha^2 \qquad i = j \\ \langle a_i b_j \rangle &= 0 \qquad \text{for all } i \text{ and } j \end{aligned} \tag{A2}$$

where α^2 is a constant.[10] Taking the Fourier transform of (A1), $N'_T(\omega)$ is obtained;

$$\begin{aligned} N'_T(\omega) &= \int_{-T/2}^{T/2} n'(t) e^{-j\omega t}\, dt \\ &= \sum_{i=0}^{N} \left(a_i \int_{-T/2}^{T/2} \cos i\omega_0 t e^{-j\omega t}\, dt + b_i \int_{-T/2}^{T/2} \sin i\omega_0 t e^{-j\omega t}\, dt \right) \\ &= \sum_{i=0}^{N} \frac{a_i T}{2} \left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} + \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2} \right] - \frac{jb_i T}{2} \left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} - \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2} \right] \end{aligned} \tag{A3}$$

The probability density of the real and imaginary parts of $N'_T(\omega)$ will now be found. Since a_i is gaussian and according to (A2) a_i and a_j are uncorrelated for $i \neq j$, then a_j must be statistically independent of a_i for $i \neq j$.[11] Hence, the real part of (A3) is a sum of independent gaussian random variables and, therefore, it too must be gaussian. Similarly, the imaginary part of (A3) is seen to be gaussian. Since the means of a_i and b_i are zero, the means of the real and imaginary parts of (A3) must also be zero. The variance of the real part of $N'_T(\omega)$ is

$$\left\langle \left| \sum_{i=0}^{N} \frac{a_i T}{2} \left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} + \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2} \right] \right|^2 \right\rangle = \frac{\alpha^2 T^2}{4} \sum_{i=0}^{N} \left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} + \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2} \right]^2 \tag{A4}$$

Similarly, the variance of the imaginary part of $N'_T(\omega)$ is seen to be

$$\frac{\alpha^2 T^2}{4} \sum_{i=0}^{N} \left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} - \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2} \right]^2 \tag{A5}$$

Now the probability density of both the real and imaginary parts of $N'_T(\omega)$ has been shown to be gaussian with mean zero and variance given by (A4) and (A5), respectively.

Note now that variance of the real part of $N'_T(\omega)$ is equal to the variance of the imaginary part at values of ω which are multiples of $2\pi/T$. At these values, the variance is equal to $\alpha^2 T^2/4$. Secondly, the real part of $N'_T(\omega)$ can be seen to be uncorrelated with the imaginary part, since $\langle a_i b_j \rangle = 0$. Therefore, the real part is statistically independent of the imaginary part since uncorrelated gaussian random variables are independent.

An expression for α^2 in terms of $Q_{n'}$, and T will now be found. By definition from (A1)

$$\begin{aligned} \alpha^2 = \langle |a_i|^2 \rangle &= \left\langle \left| \frac{2}{T} \int_{-T/2}^{T/2} n'(t) \cos i\omega_0 t dt \right|^2 \right\rangle \\ &= \frac{4}{T^2} \iint_{-T/2}^{T/2} \langle n'(t_1) n'(t_2) \rangle \cos i\omega_0 t_1 \cos i\omega_0 t_2 dt_1 dt_2 \end{aligned} \tag{A6}$$

In order to evaluate (A6), let us assume that $n'(t)$ has approximately the same autocorrelation as white noise. Then it follows that

$$\langle n'(t_1) n'(t_2) \rangle = \delta(t_1 - t_2) Q_{n'} \tag{A7}$$

where $Q_{n'}$ is the area under the autocorrelation function of $n'(t)$.

Substituting (A7) into (A6), we obtain

$$\alpha^2 = \frac{4Q_{n'}}{T^2} \int_{-T/2}^{T/2} \cos^2 i\omega_0 t dt = \frac{2Q_{n'}}{T} \tag{A8}$$

Hence, at multiples of $2\pi/T$, the variance of the real and imaginary parts of $N'_T(\omega)$ is $Q_{n'}T/2$.

Now the real and imaginary parts of $N'_T(\omega)$ have been shown to be statistically independent of each other. Both parts have also been shown to be gaussian with mean zero and variance equal to $Q_{n'}T/2$ at multiples of $2\pi/T$. Hence, it follows that $N'_T(\omega)$ at multiples in ω of $2\pi/T$ is Rayleigh distributed in amplitude[12]

$$p[|N'_T(\omega)|] = [2|N'_T(\omega)|/TQ_{n'}] e^{-|N'_T(\omega)|^2/Q_{n'}T} \tag{A9}$$

and uniformly distributed in phase

$$p[\angle N'_T(\omega)] = 1/2\pi \qquad 0 \leq \angle N'_T(\omega) \leq 2\pi$$

In a similar manner, it can be shown that

$$p[|X_T(\omega)|] = [2|X_T(\omega)|/TQ_x] e^{-|X_T(\omega)|^2/Q_x T} \tag{A10}$$

and

$$p[\angle X_T(\omega)] = 1/2\pi \quad 0 \leq \angle X_T(\omega) \leq 2\pi$$

It should be noted that the preceding expressions are exactly true only when ω is a multiple of $2\pi/T$ since the variances of the real and imaginary parts of (A3) are only equal at these values of ω.

Appendix B: Derivation of the Probability Density of $E'_T(\omega)$

It has been shown in Appendix A that the probability density of $N'_T(\omega)$ is

$$\begin{aligned} p[|N'_T(\omega)|] &= [2|N'_T(\omega)|/Q_{n'}T] e^{-|N'_T(\omega)|^2/Q_{n'}T} \\ p[\angle N'_T(\omega)] &= 1/2\pi \end{aligned} \tag{B1}$$

and that the probability density of $X_T(\omega)$ is

$$\begin{aligned} p[|X_T(\omega)|] &= [2|X_T(\omega)|/Q_x T] e^{-|X_T(\omega)|^2 Q_x T} \\ p[\angle X_T(\omega)] &= 1/2\pi \end{aligned} \tag{B2}$$

Now the probability density of $E'_T(\omega)$ will be found.

The probability density of the amplitude of $E'_T(\omega)$ will be found first. It is shown that[13]

$$p[|E'_T(\omega)|] = \int_0^\infty |X_T(\omega)| p[|X_T(\omega)|| E'_T(\omega)|, |X_T(\omega)|] d|X_T(\omega)| \tag{B3}$$

where $p[|N'_T(\omega)|, |X_T(\omega)|]$ is the joint probability density of $|N'_T(\omega)|$ and $|X_T(\omega)|$. Since $|N'_T(\omega)|$ and $|X_T(\omega)|$ has been assumed independent, then

$$p[|N'_T(\omega)|, |X_T(\omega)|] = p[|N'_T(\omega)|] p[|X_T(\omega)|] \tag{B4}$$

Substituting (B1, B2, and B4) into (B3) and simplifying, one obtains

$$p[|E'_T(\omega)|] = \frac{4|E'_T(\omega)|}{Q_x Q_{n'} T^2} \int_0^\infty |X_T(\omega)|^3 e^{-|E'T(\omega)|^2/Q_x T} \times \exp\left[\frac{-|E'_T(\omega)|^2 |X_T(\omega)|^2}{Q_{n'}T} \right] d|X_T(\omega)| \tag{B5}$$

Equation (B5) can be integrated by parts to yield

$$p[|E'_T(\omega)|] = \frac{|E'_T(\omega)|Q_{n'}}{2Q_x}\left[\frac{1}{|E'_T(\omega)|^2/2 + Q_{n'}/2Q_x}\right]^2 \quad \text{(B6)}$$

Noting that the phase of $E'_T(\omega)$ is the difference of the phase of $N'_T(\omega)$ and $X_T(\omega)$, the probability density of the phase of $E'_T(\omega)$ is found by convolving $p[\angle N'_T(\omega)]$ with the $p[-\angle X_T(\omega)]$. Since the phases of $N'_T(\omega)$ and $X_T(\omega)$ are statistically independent of each other and uniformly distributed from zero to 2π, it can readily be shown that the probability density of the phase of $E'_T(\omega)$ is triangular and given by

$$\begin{aligned} p(\theta) &= -(1/4\pi^2)\theta + 1/2\pi & 0 \le \theta \le 2\pi \\ &= (1/4\pi^2)\theta + 1/2\pi & -2\pi \le \theta \le 0 \\ &= 0 & |\theta| > 2\pi \end{aligned} \quad \text{(B7)}$$

where $\theta = \angle E'_T(\omega)$. However, this particular triangular probability density is also equivalent to a uniform density from zero to 2π due to the ambiguity of phase. Since it is impossible to distinguish between a particular phase angle θ and $\theta - 2\pi$, we can add the probability densities of θ and $\theta - 2\pi$ to obtain the total probability density of the particular phase angle occurring. It then follows that

$$\begin{aligned} p(\theta) + p(\theta - 2\pi) &= -\theta^2/4\pi^2 + 1/2\pi + \\ &\qquad (1/4\pi^2)(\theta - 2\pi) + 1/2\pi \\ &= 1/2\pi \quad 0 \le \theta \le 2\pi \end{aligned}$$

Hence the probability density of $\angle E'_T(\omega)$ is uniform between zero and 2π with a value of $1/2\pi$.

The mean and variance of the probability density of $E'_T(\omega)$ will now be derived. The mean is seen to be

$$\begin{aligned} \langle E'_T(\omega)\rangle &= \langle |E'_T(\omega)| \cos\theta\rangle + j\,\langle |E'_T(\omega)| \sin\theta\rangle \\ &= \langle |E'_T(\omega)|\rangle\langle\cos\theta\rangle + j\langle|E'_T(\omega)|\rangle\langle\sin\theta\rangle \end{aligned} \quad \text{(B8)}$$

where $\theta = \angle E'_T(\omega)$. Since θ is uniformly distributed between 0 and 2π, both $\langle\cos\theta\rangle$ and $\langle\sin\theta\rangle$ are zero. Hence

$$\langle E'_T(\omega)\rangle = 0$$

The variance of $E'_T(\omega)$ is given by

$$\sigma^2[E'_T(\omega)] = \int_0^\infty \frac{|E'_T(\omega)|^3 d|E'_T(\omega)|}{2Q_xQ_{n'}[|E'_T(\omega)|^2/2Q_{n'} + 1/2Q_x]^2} \quad \text{(B9)}$$

Equation (B8) can be reduced to the following form with the aid of integral tables[14]:

$$\sigma^2[E'_T(\omega)] = \left[\int_0^\infty \frac{|E'_T(\omega)|\, d|E'_T(\omega)|}{Q_{n'}/Q_x + |E'_T(\omega)|^2} - \frac{Q_{n'}}{Q_x}\int_0^\infty \frac{|E'_T(\omega)|d|E'_T(\omega)|}{[Q_{n'}/Q_x + |E'_T(\omega)|^2]^2}\right]\frac{2Q_{n'}}{Q_x} \quad \text{(B10)}$$

The first integral in (B10) diverges while the second converges, and hence the variance of $E'_T(\omega)$ is infinity or undefined.

The preceding derivation did not take into account the practical fact that $|E'_T(\omega)|$ must be less than some fixed value. As explained earlier in the main discussion, the only way by which $|E'_T(\omega)|$ can be arbitrarily large is for $N'_T(\omega)$ to be arbitrarily large or $X_T(\omega)$ to be arbitrarily small. The former is ruled out by the limited dynamic range of the recording media and the latter ruled out, since estimates of $H(\omega)$ are disregarded when $|X_T(\omega)|$ is small. Therefore, in reality the probability density of $|E'_T(\omega)|$ is given by

$$p[|E'_T(\omega)|] = \frac{KQ_{n'}\,|E'_T(\omega)|}{2Q_x[|E'_T(\omega)|^2/2 + Q_{n'}/2Q_x]^2} \quad \text{(B11)}$$

$$\text{for } |E'_T(\omega)| \le C$$

$$\text{and } p(|E'_T(\omega)|) = 0 \text{ for } |E'_T(\omega)| > C$$

The constant K is chosen to keep the area under the probability density equal to unity and C is the maximum value of $|E'_T(\omega)|$. Since the derivations in the report do not require C and K to be determined, no attempt will be made to do so. Since $|E'_T(\omega)|$ has been truncated at C, the probability density now has a variance which will be denoted by σ'^2.

The probability density of $E'_T(\omega)$ is a special case of a more general probability density known as the complex Wishart probability density. This more general probability density can be shown to reduce to Eq. (B6).[15]

Appendix C: Derivation of the Probability Density of $E''_T(\omega)$

The error term $E''_T(\omega)$ is defined as

$$E''_T(\omega) = N''_T(\omega)/X_T(\omega) \quad \text{(C1)}$$

which is a deterministic function $N''_T(\omega)$ divided by a random variable $X_T(\omega)$. In Appendix A, $X_T(\omega)$ is shown to be Rayleigh distributed in amplitude

$$p[|X_T(\omega)|] = [2|X_T(\omega)|/Q_xT]e^{-|X_T(\omega)|^2/Q_xT} \quad \text{(C2)}$$

and uniformly distributed in phase;

$$p[\angle X_T(\omega)] = 1/2\pi$$

It follows[16] that the probability density of the amplitude and phase of $E''_T(\omega)$ is given by

$$p[|E''_T(\omega)|] = \frac{2|N''_T(\omega)|^2}{Q_xT|E''_T(\omega)|^3}\exp\left[\frac{-|N''_T(\omega)|^2}{Q_xT|E''_T(\omega)|^2}\right] \quad \text{(C3)}$$

$$p[\angle E''_T(\omega)] = 1/2\pi$$

Since $E''_T(\omega)$ is uniformly distributed in phase, the mean of $E''_T(\omega)$ is zero.

The variance of $E''_T(\omega)$ is given by

$$E[|E''_T(\omega)|^2] = 2\int_0^\infty \frac{|N''_T(\omega)|^2}{Q_xT\gamma} e^{-|N''_T(\omega)|^2/Q_xT\gamma^2}\, d\gamma \quad \text{(C4)}$$

where $\gamma = |E''_T(\omega)|$ in the integrand. Using the substitution $y = 1/\gamma$, this integral reduces to

$$E[|E''_T(\omega)|^2] = 2\int_0^\infty \frac{|N''_T(\omega)|^2}{Q_xTy} e^{-|N''_T(\omega)|^2y^2/Q_xT}\, dy \quad \text{(C5)}$$

Now let

$$\xi = |N''_T(\omega)|^2/Q_xT$$

then Eq. C5 becomes

$$\begin{aligned} E[|E''_T(\omega)|^2] &= 2\int_0^\infty \frac{\xi}{y} e^{-\xi y^2}\, dy \\ &= 2\xi\left\{\ln \xi^{1/2}y\Big]_0^1 + \sum_{n=1}^\infty \frac{(-1)^n(\xi^{1/2}y)^{2n}}{(n!)(2n)}\Big]_0^1\right\} \\ &\qquad + 2\int_1^\infty \frac{\xi}{y} e^{-\xi y^2}\, dy \end{aligned}$$

which diverges since the log of zero is undefined. But the only situation for which $y = 0$ will be when $X_T(\omega) = 0$ or $N''_T(\omega) = \infty$. As pointed out in the discussion of $E'_T(\omega)$, values of $X_T(\omega)$ which are very small are disregarded and the dynamic range of the recording media limits the observed magnitude of $N''_T(\omega)$. Hence y in (C5) is never zero and the term $E''_T(\omega)$ has a variance denoted by σ''^2.

Appendix D: Proof that $\bar{E}_T(\omega)$ is Statistically Independent at Multiples in ω of $2\pi/T$

In the case of stationary inputs and responses $\bar{E}_T(\omega)$ can be written as

$$\begin{aligned}\bar{E}_T(\omega) &= E'_T(\omega) + E''_T(\omega)\\ &= N'_T(\omega)/X_T(\omega) + N''_T(\omega)/X_T(\omega)\end{aligned}$$

$E'_T(\omega)$ will first be shown independent at multiples of $2\pi/T$, then the same will be shown for $E''_T(\omega)$.

It has been assumed that $N'_T(\omega)$ is independent of $X_T(\omega)$. To prove that $E'_T(\omega)$ is independent at multiples in ω of $2\pi/T$, it remains to be shown that $N'_T(\omega)$ is independent of itself in multiples in ω of $2\pi/T$ and that $X_T(\omega)$ is independent of itself at multiples in ω of $2\pi/T$.

In Eq. (A3) of Appendix A, the following expression was derived for $N'_T(\omega)$:

$$N'_T(\omega) = \sum_{i=0}^{N} \frac{a_i T}{2}\left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} + \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2}\right] - \frac{jb_i T}{2}\left[\frac{\sin(\omega - i\omega_0)T/2}{(\omega - i\omega_0)T/2} - \frac{\sin(\omega + i\omega_0)T/2}{(\omega + i\omega_0)T/2}\right] \quad \text{(D1)}$$

$N'_T(\omega)$ is summation of terms of the form $\sin x/x$ centered at frequency multiples of $2\pi/T$. Since uncorrelated gaussian random variables are independent, it follows from Eq. (A2) of Appendix A that the coefficients of the $\sin x/x$ terms centered in the interval zero to $2\pi N/T$ are also independent.

Note that at multiples of $2\pi/T$, the only contribution to $N'_T(\omega)$ is made by the main lobe of the $\sin x/x$ function centered at that frequency. Since the height of the main lobes are independent at multiples of $2\pi/T$, it follows that $N'_T(\omega)$ is independent at multiples of $2\pi/T$ from 0 to $2\pi N/T$. A similar argument will show that the values of $X_T(\omega)$ are independent at multiples of $2\pi/T$. Hence, values of $E'_T(\omega)$ can be concluded to be independent at multiples of $2\pi/T$.

$X_T(\omega)$ is independent at multiples of $2\pi/T$. It immediately follows that a $E''_T(\omega)$, which is composed of the deterministic $N''_T(\omega)$ divided by $X_T(\omega)$, is also independent at multiples of $2\pi/T$. In the transient case,

$$\begin{aligned}\bar{E}_T(\omega) &= E'_{T,t}(\omega)\\ &= N'_T(\omega)/X_{T,t}(\omega)\end{aligned}$$

$\bar{E}_T(\omega)$ is composed of the random variable $N'_T(\omega)$ divided by the deterministic $X_{T,t}(\omega)$. Since $N'_T(\omega)$ is independent at multiples of $2\pi/T$, $\bar{E}_T(\omega)$ must be independent at those same frequencies.

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